

Lower Bounds on Testing Functions of Low Fourier Degree

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Abstract

We consider the problem of testing whether a Boolean function has Fourier degree $\leq k$ or it is ϵ -far from any Boolean function with Fourier degree $\leq k$, we improve the known lower bound of $\Omega(k)$ [4, 6], to $\Omega(k/\sqrt{\epsilon})$. The lower bound uses the recently discovered connections between property testing and communication complexity by Blais *et. al.* [4]

1 Introduction

Following formulation of Property Testing of functions was suggested in [10]:

Let P be a fixed property of functions, and f be an unknown function. The goal is to determine (possibly probabilistically) if f has property P or if it is far from any function with property P , where distance between functions is measured with respect to some distribution D on the domain of f . More precisely $\text{Dist}(f, g) = \Pr_D(f(x) \neq g(x))$. Towards this end, one is given examples of the form $(x, f(x))$, where x is distributed according to D . One may also be allowed to query f on instances of one's choice.

The above formulation is inspired by the PAC learning model [15]. In fact, property testing is related to variants of PAC learning as has been shown in [10]. The above formulation allows defining the distance measure according to arbitrary distributions over the domain, it also allows defining property testing problems in which testers observe only randomly chosen instances (rather than being able to query instances of their own choice).

The concept of property testing was introduced in the context of program checking by Blum, Luby and Rubinfeld [5] who showed that *linearity* of a function over a vector

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space can be tested with a *constant* number of queries. A central ingredient in the proof of the $\text{MIP} = \text{NEXP}$ theorem [1] was the proof that *multilinearity* can be tested with a *polylogarithmic* number of queries.

Over the course of this effort, a variety of techniques have been developed for designing property testing algorithms thus proving testing upper bounds. However, as is often the case in theoretical computer science, lower bounds are harder to come by. Although several lower bounds for known problems are known, until very recently few general techniques were known beyond the use of Yao's minimax lemma. Blais et. al. [4] came up with a new technique to prove property testing lower bounds, by using known lower bounds for randomized communication complexity problems. In particular, they show how to reduce certain communication complexity problems to testing problems, thus showing that communication lower bounds imply lower bounds for property testing. They show that this technique is indeed powerful by applying it on many testing problems and improving on some previous known lower bounds for testing k -linearity, k -juntas, Fourier degree $\leq k$, s -sparse $GF(2)$ -polynomials, etc. It has not been obvious to come up with lower bounds with dependence on the distance parameter ϵ using this technique. In Theorem 1.1, we show that this technique can be used to prove a lower bound on testing the property of having lower Fourier degree, which is related to the distance parameter ϵ .

For property testing problems, it is natural to ask what happens to the testing complexity of the problem if we consider relaxations of the original problem. Fischer et al. [9], noticed that the complexity of one of their algorithms for testing k -juntas can be improved to have a quadratic dependence on k if the algorithm is only required to reject functions that are far from being $2k$ -juntas. This relaxation was also considered by Blais et al. [4]. They applied their communication complexity technique to prove a lower bound of $\Omega(\min\{\binom{k}{t}^2, k\} - \log k)$ on the number of queries necessary for distinguishing between functions that are k -juntas and functions that are ϵ -far from $(k+t)$ -juntas. This of course does not give a good lower bound for $t = \delta k$ when δ is a constant. Ron and Tsur [14] in a recent paper study the problem of testing whether a function is a k -junta or ϵ -far from a $(1 + \delta)k$ -junta. They give a $O(\frac{k \log(1/\delta)}{\epsilon \delta^2})$ upper bound and a $\Omega(k/\log k)$ lower bound for the case when $\epsilon = O(1/\log k)$.

Upper bounds of $2^{O(d)}$ have been given on the general problem of testing whether a Boolean function has Fourier degree $\leq d$ or is ϵ -far from any Boolean function with Fourier degree d [7, 8]. The best lower bounds known on this problem is $\Omega(d)$ [4, 6], which holds for any $\epsilon \leq 1/2$. In this paper we show a lower bound of $\Omega(d/\sqrt{\epsilon})$ for this problem.

Theorem 1.1 (Main Theorem). *Let $\epsilon \geq 2^{-k-1}$. Testing whether a Boolean function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ has Fourier degree $\leq k$ or ϵ -far from any Boolean function with Fourier degree $\leq k + 1$ requires $\Omega(\frac{k}{\sqrt{\epsilon}})$ queries.*

2 Fourier Degree at most d

In this section we present known results for the problem of testing whether a given Boolean function has low Fourier degree. For convenience, in the context of Fourier analysis we consider the Boolean function to be of the form $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$.

Definition 2.1. *It is a well known fact that for the set of characters $\chi_S = \prod_{i \in S} x_i$, every Boolean function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ has a unique representation in the form*

$$f = \sum_{S \subseteq [n]} \chi_S \hat{f}(S),$$

where $\hat{f}(S) \in \mathbb{R}$ are called the Fourier coefficients of f . The Fourier degree of a Boolean function f is equal to the maximum $k \geq 0$ such that $\hat{f}(S) \neq 0$ for a set S of size k .

Diakonikolas *et al.* [8] considered the problem of testing whether a Boolean function f has Fourier degree at most d . They proved a general lower bound of $\tilde{\Omega}(\log d)$, and a lower bound of $\tilde{\Omega}(\sqrt{d})$ for the non-adaptive tester and any $\epsilon \leq 1/2$. They also present an algorithm with $\tilde{O}(2^{6d}/\epsilon^2)$ query complexity for this problem. Chakraborty *et al.* [6] and later Blais *et al.* [4] improved the lower bound to $\Omega(d)$, for any $0 \leq \epsilon \leq 1/2$. We present the lower bound given by Blais *et al.* in Section 3.3.

3 Lower Bounds Via Communication Complexity

3.1 Communication Complexity

In this section we will give a brief introduction to Communication Complexity, and state known lower bounds for the famous set disjointness problem. The two-party communication model was introduced by Andrew Chi-Chih Yao [16] in 1979. In this model, two parties, traditionally called Alice and Bob, are trying to collaboratively compute a known Boolean function $F : X \times Y \rightarrow \{0, 1\}$. Each party is computationally unbounded; however, Alice is only given input $x \in X$ and Bob is only given $y \in Y$. In order to compute $F(x, y)$, Alice and Bob communicate in accordance with an agreed-upon *communication protocol* \mathcal{P} . Protocol \mathcal{P} specifies as a function of transmitted bits only whether the communication is over and, if not, who sends the next bit. Moreover, \mathcal{P} specifies as a function of the transmitted bits and x the value of the next bit to be sent by Alice. Similarly for Bob. The communication is over as soon as one of the parties knows the value of $F(x, y)$. The cost of the protocol \mathcal{P} is the number of bits exchanged on the worst input.

Definition 3.1. *The deterministic communication complexity of F , denoted by $DC(F)$, is the cost of an optimal communication protocol computing F .*

There are several ways in which the deterministic communication model can be extended to include randomization. In the *public-coin* model, Alice and Bob have access to a shared random string r chosen according to some probability distribution. The only difference in the definition of a protocol is that now the protocol \mathcal{P} specifies the next bit to be sent by Alice as a function of x , the already transmitted bits, and a random string r . Similarly for Bob. In the *private-coin* model, Alice has access to a random string r_A hidden from Bob, and Bob has access to a random string r_B hidden from Alice.

Definition 3.2. *The bounded-error randomized communication complexity of F with public coins (private coins), denoted by $\text{RC}_2(F)$ ($\text{RC}_2^{\text{pri}}(F)$), is the minimum cost of a public-coin (private-coin) randomized protocol that computes F correctly with probability at least $2/3$ on every input. (The subscript 2 refers to permitting 2-sided error.)*

Clearly, for every Boolean F we have $\text{RC}_2(F) \leq \text{RC}_2^{\text{pri}}(F)$. Ilan Newman [12] showed that the two measures are identical up to constant multiplicative factors and logarithmic additive terms.

Theorem 3.3 (Newman [12]). *For every Boolean function $F : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ we have*

$$\text{RC}_2^{\text{pri}}(F) = O(\text{RC}_2 + \log n).$$

Set Disjointness

Alice and Bob are given x and y , $x, y \in \{-1, 1\}^n$, and compute

$$\text{DISJ}(x, y) = \bigvee_{i=1}^n (x_i \wedge y_i),$$

where $(a \wedge b) = 1$ if $a = b = 1$, and -1 otherwise.

It is well-known that $\text{RC}_2(\text{DISJ}^n) = \Omega(n)$. The problem DISJ_k^n is a balanced version of DISJ^n with the promise that $|x| = |y| = k$, where $|x|$ is equal to the number of 1s in x , and that $x_i \wedge y_i = 1$ for at most one i . A first lower bound of $\Omega(\sqrt{n})$ when $k = n/3$ was proved by Babai et al. [2]. This bound was strengthened to $\Omega(k)$, by Kalyanasundaram and Schnitger [11], simplified by Razborov [13], and further simplified by Bar-Yossef et al. [3].

3.2 Fourier Analysis of Boolean Functions

Consider the 2^n -dimensional vector space of all functions $f : \{-1, 1\}^n \rightarrow \mathbb{R}$. An inner product on this space can be defined as follows

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \{-1, 1\}^n} f(x)g(x) = \mathbf{E}[f \cdot g],$$

where the latter expectation is taken uniformly over all $x \in \{-1, 1\}^n$. This defines the l_2 -norm

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\mathbf{E}[f^2]}.$$

Definition 3.4. For $S \subseteq [n]$, the character $\chi_S : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is defined as

$$\chi_S(x) = \prod_{i \in S} x_i.$$

The set of characters forms an orthonormal basis for the inner product space. Hence, every function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ can be written uniquely as

$$f = \sum_S \langle f, \chi_S \rangle \chi_S.$$

The above equation is referred to as the Fourier expansion of f , and the Fourier coefficient of f corresponding to set S is defined as

$$\hat{f}(S) = \langle f, \chi_S \rangle.$$

Definition 3.5. The Fourier degree of a Boolean function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is equal to maximum $k > 0$ such that there exists $S \subseteq [n]$, $|S| = k$, for which $\hat{f}(S) \neq 0$.

3.3 Reduction from Communication Complexity

In this section we present the approach which was introduced recently by Blais *et al.* [4], to prove property testing lower bounds via communication complexity. Blais *et al.* give a simple scheme for reducing communication problems to testing problems, which allows them to use known lower bounds in communication complexity to prove lower bounds in testing.

3.3.1 From Property Testing to Communication Complexity

In this section we present the general reduction introduced by Blais *et al.*, in a slightly different way. Namely, the reduction from a communication complexity problem, sometimes helps prove lower bound for testing not a property, but a property testing problem where we want to decide whether the function belongs to property F , or property G . For two families of functions \mathcal{F} and \mathcal{G} , let $P(\mathcal{F}, \mathcal{G})$ be the testing problem of determining whether a given function belongs to \mathcal{F} or \mathcal{G} . Let us denote by $Q(P)$, the query complexity of a property testing problem P , where the tester is allowed to make two-sided error up to $1/3$.

Given a testing problem $P(\mathcal{F}, \mathcal{G})$, Boolean functions $f, g : \{-1, 1\}^n \rightarrow \{-1, 1\}$, and a “combining” function $h = h(f, g)$, define the following communication game $C_{h,P}$:

Alice knows f and Bob knows g , and their goal is to decide if h belongs to \mathcal{F} or it is in \mathcal{G} .

Lemma 3.6 ([4], rephrased and partially stated). *For any function h , properties \mathcal{F} and \mathcal{G} ,*

1. $\text{RC}(C_{h,P(\mathcal{F},\mathcal{G})}) \leq 2Q(P(\mathcal{F},\mathcal{G}))$,
2. $\text{RC}^1(C_{h,P(\mathcal{F},\mathcal{G})}) \leq 2Q^1(P(\mathcal{F},\mathcal{G}))$,

where RC^1 and Q^1 are one sided error complexities.

Proof. The proof follows by showing how to use a t -query testing algorithm that distinguishes between \mathcal{F} and \mathcal{G} to create a communication protocol for $C_{h,P(\mathcal{F},\mathcal{G})}$. Alice and Bob can use public randomness to generate the required queries of the testing algorithm. For a query to $h(x)$, Alice can compute $f(x)$, and Bob can compute $g(y)$. Now they can communicate $f(x)$ and $g(y)$, and Bob can compute $h(f(x), g(y))$ and answer to the query of the tester with the value $h(f(x), g(y))$. After t queries, there has been $2t$ bits of communication, and they can use the decision of the tester on whether h has property \mathcal{F} or it has property \mathcal{G} .

The proof of (2) is analogous. □

4 Results

4.1 Our Constructions of Functions

In this section we give a method how to construct functions which are of Fourier degree $\leq k$ and how to construct functions which are far from having Fourier degree at most k .

Let l be a positive integer. For any $S \subseteq [n]$, $|S| = l$, let $C_{a_1, a_2, \dots, a_l}^S$ for any $(a_1, \dots, a_l) \in \{-1, 1\}^l$ be a subset of $[n] \setminus S$.

Now for any $S \subseteq [n]$ and sets $\{C_{a_1, \dots, a_l}^S\}$, let $f^S : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be the Boolean function defined as following

$$f^S(x_1, \dots, x_n) = \chi_{C_{x_S}}(x_1, \dots, x_n),$$

where $\chi_A(x_1, \dots, x_n) = \prod_{i \in A} x_i$ for $A \subseteq [n]$.

In the next two propositions we show how cardinalities of sets C_{a_1, \dots, a_l} can lead f^S to be of low Fourier degree, or to be far from any Boolean function with Fourier degree k .

Proposition 4.1. *The Boolean function $f^{[l]} : \{-1, 1\}^n \rightarrow \{-1, 1\}$ described above, is of Fourier degree $m + l$ if*

$$\forall (a_1, \dots, a_l) \in \{-1, 1\}^l, |C_{a_1, \dots, a_l}^{[l]}| \leq m,$$

where $[l] = \{1, 2, \dots, l\}$.

Proof.

For the sake of simplicity we will not write the superscript $[l]$ for $C_{a_1, \dots, a_l}^{[l]}$. We have to prove that $\langle f, \chi_S \rangle = 0$ for any $S \subseteq [n]$ with $|S| \geq m + l + 1$.

$$\begin{aligned} \widehat{f^{[l]}}(S) &= \langle f^{[l]}, \chi_S \rangle = 2^{-n} \sum_{x \in \{-1, 1\}^n} \chi_{C_{x_{[l]}}}(x) \cdot \chi_S(x) \\ &= 2^{-n} \sum_{x_1, \dots, x_l} \sum_{x_{l+1}, \dots, x_n} \chi_{C_{x_1, \dots, x_l}}(x_1, \dots, x_n) \cdot \chi_S(x_1, \dots, x_n) = 0. \end{aligned}$$

The last equality follows from the fact that

$$\sum_{x_{l+1}, \dots, x_n \in \{-1, 1\}} \chi_{C_{x_1, \dots, x_l}}(x_1, \dots, x_n) \cdot \chi_S(x_1, \dots, x_n) = 0,$$

since

$$\exists i \in S : i \notin C_{x_1, \dots, x_l} \cup \{1, \dots, l\},$$

because $|S| \geq m + l + 1$. □

Proposition 4.2. *The Boolean function $f^{[l]}$ is $1/2^{l+1}$ -far from any Boolean function of Fourier degree $m - 1$ if for only one $(b_1, \dots, b_l) \in \{-1, 1\}^l$, $|C_{(b_1, \dots, b_l)}| \geq m$, and*

$$\forall (a_1, \dots, a_l) \neq (b_1, \dots, b_l) : |C_{a_1, \dots, a_l}| \leq m - 1.$$

Proof. First we prove that for any $U \subseteq \{1, \dots, l\}$, the fourier coefficient of $|f^{[l]}|$ at $S = U \cup C_{b_1, \dots, b_l}$ is equal to $1/2^l$.

$$\begin{aligned} \widehat{f^{[l]}}(S) &= \langle f^{[l]}, \chi_{U \cup C_{b_1, \dots, b_l}} \rangle \\ &= 2^{-n} \sum_{x \in \{-1, 1\}^n} \chi_{x_{[l]}}(x) \cdot \chi_{U \cup C_{b_1, \dots, b_l}}(x) \\ &= 2^{-n} \sum_{x_{[l]} \in \{-1, 1\}^l} \left(\prod_{i \in U} x_i \right) \sum_{x_{l+1}, \dots, x_n} \chi_{x_{[l]}}(x) \chi_{C_{b_1, \dots, b_l}}(x) \\ &= 2^{-l} \prod_{i \in U} b_i \\ &\quad + 2^{-n} \sum_{x_{[l]} \neq (b_1, \dots, b_l)} \left(\prod_{i \in U} x_i \right) \sum_{x_{l+1}, \dots, x_n} \chi_{x_{[l]}}(x) \cdot \chi_{C_{b_1, \dots, b_l}}(x) \\ &= 2^{-l} \prod_{i \in U} b_i. \end{aligned}$$

The last equality follows from the fact that if $(a_1, \dots, a_l) \neq (b_1, \dots, b_l)$ then $|C_{(b_1, \dots, b_l)}| > |C_{(a_1, \dots, a_l)}|$, therefore

$$\sum_{x_{l+1}, \dots, x_n} \chi_{x_{[l]}}(x) \cdot \chi_{C_{b_1, \dots, b_l}}(x) = 0.$$

Let $g : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be a Boolean function with fourier degree $m - 1$, thus we can write

$$g(x) = \sum_{\substack{|S| \leq m-1 \\ S \subseteq [n]}} \widehat{g}(S) \chi_S(x).$$

Notice that the distance between two functions f and g with range $\{-1, 1\}$ can be written as $\frac{1}{2} \|f - g\|_2^2 = \frac{1}{2} \mathbf{E}[(f - g)^2]$. Finally Parseval's identity implies that

$$\begin{aligned} \|f - g\|_2^2 &= \sum_{\substack{|S| \leq m-1 \\ S \subseteq [n]}} (\widehat{f}(S) - \widehat{g}(S))^2 + \sum_{\substack{|S| \geq m \\ S \subseteq [n]}} \widehat{f}(S)^2 \\ &\geq \sum_{\substack{|S| \geq m \\ S \subseteq [n]}} \widehat{f}(S)^2 \geq \sum_{U \subseteq [l]} \left(2^{-l} \prod_{i \in U} b_i \right)^2 = 2^{-l}. \end{aligned}$$

□

Proposition 4.3. *The Boolean function $f^{[l]}$ is $1/2^{2l+1}$ -far from any Boolean function of Fourier degree $m + l - 1$ if for only one $(b_1, \dots, b_l) \in \{-1, 1\}^l$, $|C_{b_1, \dots, b_l}| \geq m$, and*

$$\forall (a_1, \dots, a_l) \neq (b_1, \dots, b_l) : |C_{(a_1, \dots, a_l)}| \leq m - 1.$$

Proof. The proof is similar to the proof of Proposition 4.2, with the difference that we only use the fact that $\widehat{f^{[l]}}(\{1, \dots, l\} \cup C_{b_1, \dots, b_l}) = 2^{-l}$, and thus f is 2^{-2l-1} far from any function with fourier degree $m + l - 1$. □

4.2 OR of disjoint copies of DISJ_k^m

In this section we present a new communication problem which later will be used to prove our lower bounds.

$\text{DISJ}_k^{l,m}$: Alice and Bob are given Boolean strings of length lm , where $x, y \subseteq \{-1, 1\}^{lm}$, with the extra promise that for every i ,

$$|x_{\{(i-1)m+1, \dots, im\}}| = |y_{\{(i-1)m+1, \dots, im\}}| = k,$$

and $x_j \wedge y_j = 1$ for at most one $j \in \{(i-1)m+1, \dots, im\}$. The goal is to compute

$$\bigvee_i \text{DISJ}_k(x_{\{(i-1)m+1, \dots, im\}}, y_{\{(i-1)m+1, \dots, im\}}).$$

$\text{DISJ}_k^{l,m}$ is basically OR of l disjoint copies of DISJ_k^m .

Lemma 4.4. $\text{RC}(\text{DISJ}_k^{l,m}) = \Omega(\min\{lk, l(m-k)\})$.

Proof. Assume that $k \leq m/2$. The case when k is large can be reduced to the $k \leq m/2$ case by fixing a number of elements in both sets and decreasing the size of the universal set. We reduce $\text{DISJ}_{lk/4}^{lk}$ to $\text{DISJ}_k^{l,m}$. The reduction is very simple. Assume there is a protocol \mathcal{P} for solving $\text{DISJ}_k^{l,m}$. Alice and Bob are given strings x and y respectively, of length lk , and want to decide whether they are disjoint or not. They are also promised that $x_i = y_i = 1$ for at most one choice of i . Let both Alice and Bob divide x and y to l strings of length k , $x = x^1 x^2 \dots x^l$ and $y = y^1 y^2 \dots y^l$. Now for each i , Alice constructs

$$x_c^i = 1^{k-|x^i|} (-1)^{m-2k+|x^i|},$$

where π^a represents concatenation of a copies of π . Bob constructs

$$y_c^i = (-1)^{m-2k+|y^i|} 1^{k-|y^i|}.$$

Finally they use protocol \mathcal{P} to solve $\text{DISJ}_k^{l,m}$ on inputs

$$x' = x^1 x_c^1 x^2 x_c^2 \dots x^l x_c^l \text{ and } y' = y^1 y_c^1 y^2 y_c^2 \dots y^l y_c^l.$$

Notice that the choice of x_c^i and y_c^i makes every block in x' and every block in y' have exactly k , 1s. Moreover, this construction preserves the property that there is only one i for which $x'_i = y'_i = 1$, and the resulting problem is equivalent to the original disjointness problem.

As a result

$$\text{RC}(\text{DISJ}_k^{l,m}) \geq \text{RC}(\text{DISJ}_{lk/4}^{lk}) = \Omega(lk).$$

□

4.3 Proof of Theorem 1.1

In this section, we show how to use the communication complexity technique to prove a lower bound of $\Omega(k/\sqrt{\epsilon})$ on testing whether a Boolean function is of Fourier degree at most k .

Proof of Theorem 1.1. Let l be the largest integer such that $\epsilon < 2^{-2l-1}$. We prove that $\Omega(k \cdot 2^l)$ queries are required to test whether a Boolean function has Fourier degree $\leq k$ or is ϵ -far from any Boolean function with degree $\leq k+1$. Notice that since $\epsilon \geq 2^{-k-1}$ thus $l \leq \frac{k}{2}$.

Let $f^{[l]} : \{-1, 1\}^n \rightarrow \{-1, 1\}$, be defined for a family of subsets of $[n]$, $\{C_{a_1, \dots, a_l}^{[l]}\}$ as explained in Section 4.1. Similarly define $g^{[l]} : \{-1, 1\}^n \rightarrow \{-1, 1\}$ for a family of sets $\{D_{a_1, \dots, a_l}^{[l]}\}$. Let $h = h(f, g) = f \cdot g \cdot \chi_{[n] \setminus [l]}$. Alice is given f and Bob is given g , and their goal, (testing problem P), is to decide whether $h(f, g)$ has Fourier degree at most k or it is ϵ -far from every Boolean function with Fourier degree at most k .

Assume that $n - k$ is even, and let $C_{h,P}$ have the extra promise that for every $(a_1, \dots, a_l) \in \{-1, 1\}^l$, $|C_{a_1, \dots, a_l}^{[l]}| = |D_{a_1, \dots, a_l}^{[l]}| = (n - k)/2$. Moreover with the promise that $|C_{a_1, \dots, a_l}^{[l]} \cap D_{a_1, \dots, a_l}^{[l]}| \leq 1$.

Notice that $C_{h,P}$ is equivalent to $\text{DISJ}_{(n-k)/2}^{2^l, n-k}$, where the i th block of the input to Alice represents $C_{a_1, \dots, a_l}^{[l]}$ and i th block of the input to Bob represents $D_{a_1, \dots, a_l}^{[l]}$, where (a_1, \dots, a_l) is the i -th vector in $\{-1, 1\}^l$ in the chronological order. By Lemma 3.6 we have

$$2Q(h) \geq \text{RC}(C_{h,P}) = \text{DISJ}_{(n-k)/2}^{2^l, n-k} = \Omega(k \cdot 2^l),$$

where the last equality follows from Lemma 4.4. Now the result follows by Proposition 4.1 and Proposition 4.3. \square

4.4 Approximate Fourier degree testing

Chakraborty *et al.* [6] proved that testing whether a Boolean function has Fourier degree at most k or it is far from any Boolean function with Fourier degree $n - \Theta(1)$ requires $\Omega(k)$ queries. Here we prove an $\Omega(1/\epsilon)$ lower bound for the non-adaptive tester, using Yao's minimax principle. For this we introduce two distributions D_p and D_n where D_p is a distribution restricted to a subset of Boolean functions with Fourier degree $\leq k$ and D_n is a distribution restricted to a subset of Boolean functions ϵ -far from any Boolean function with Fourier degree $\leq n - 2k$. Which combined with Chakraborty *et al.*'s result gives an $\Omega(k + 1/\epsilon)$ lower bound for non-adaptively approximate testing the Fourier degree.

Theorem 4.5. *Let $\epsilon \geq 1/2^{-k/2-1}$. Non-adaptively Testing whether a Boolean function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ has Fourier degree $\leq k$ or it is ϵ -far from any Boolean function with Fourier degree $\leq n - k$ requires $\Omega(\frac{1}{\epsilon} + k)$ queries.*

Proof. Let l be the largest integer such that $\epsilon < 2^{-l-1}$. We prove that $\Omega(2^l)$ queries are required to test whether a Boolean function has Fourier degree $\leq k$ or is ϵ -far from any Boolean function with degree $\leq n - k$. Notice that since $\epsilon \geq 1/2^{\frac{k}{2}-1}$ thus $l \leq \frac{k}{2} - 1$.

Let D_p be the distribution where for any $(a_1, \dots, a_l) \in \{-1, 1\}^l$ we choose uniformly at random $C_{(a_1, \dots, a_l)}$ to be a subset of size $k/2$ of $\{l + 1, \dots, n\}$. Finally constructing $f^{[l]}$ using the chosen sets as described in Section 4.1. Proposition 4.1 immediately implies that $f^{[l]}$ has Fourier degree $\leq k$.

Let D_n be the distribution where we choose $(b_1, \dots, b_l) \in \{-1, 1\}^l$ uniformly at random and choose $C_{(b_1, \dots, b_l)}$ to be a subset of cardinality $n - k + 1$ of $\{l + 1, \dots, n\}$. Also for any $(a_1, \dots, a_l) \in \{-1, 1\}^l$, where $(a_1, \dots, a_l) \neq (b_1, \dots, b_l)$, we choose uniformly at random $C_{(a_1, \dots, a_l)}$ to be a subset of cardinality $k/2$ of $\{l + 1, \dots, n\}$. Finally build $f^{[l]}$ using the chosen sets. Proposition 4.2 immediately implies that $f^{[l]}$ is 2^{-l-1} -far from any Boolean function with Fourier degree $\leq n - k$.

Let our final distribution be that with probability $1/2$ we draw $f^{[l]}$ from D_p and with probability $1/2$ we draw $f^{[l]}$ from D_n . Now by Yao's minimax principle if we prove that any deterministic algorithm that queries less than $2^l/6$ with constant probability makes a mistake, implies that the original testing problem with constant probability of error requires $\frac{2^l}{6} = \Omega(\frac{1}{\epsilon})$ queries.

For any deterministic set of $d \leq \frac{2^l}{6}$ queries to outputs of the function on inputs x^1, \dots, x^d ,

$$|\{(a_1, \dots, a_l) | (\exists 1 \leq i \leq d) x_{[l]}^i = (a_1, \dots, a_l)\}| \leq d \leq \frac{2^{l-1}}{6}.$$

Therefore the measure of the set of functions from support of D_n for which the deterministic tester has not yet queried any input from the high degree subcube is at least

$$\frac{1}{2} \cdot \frac{2^l - 2^l/6}{2^l} = \frac{5}{12} \geq \frac{1}{3}.$$

Thus with probability at least $\frac{1}{3}$ the deterministic tester will make an error. □

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